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Discrete Convex Analysis III:

Algorithms for Discrete Convex Functions

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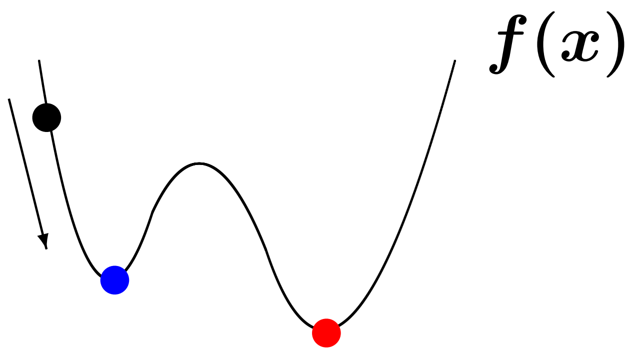
Contents of Part III

Algorithms for Discrete Convex Functions

A1.
Minimization (General)

Descent Method



S0: Initial sol x^*

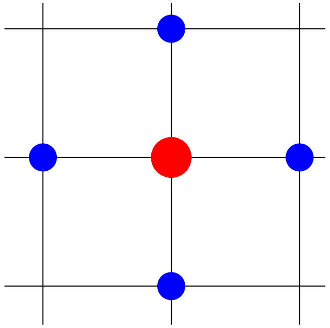
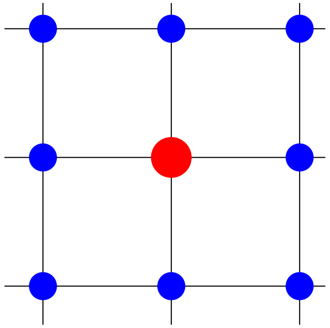
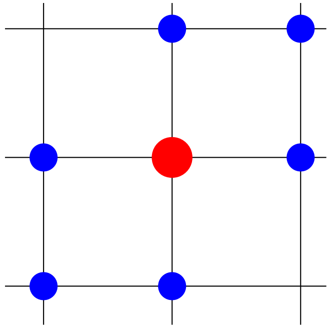
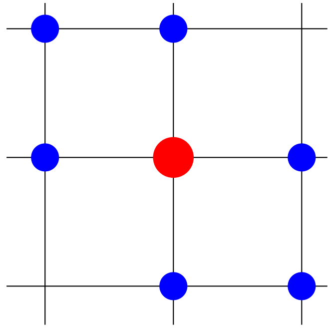
S1: Minimize $f(x)$ in nbhd of x^* to obtain $x^\bullet$

S2: If $f(x^*) \leq f(x^\bullet)$, return x^* (local opt)

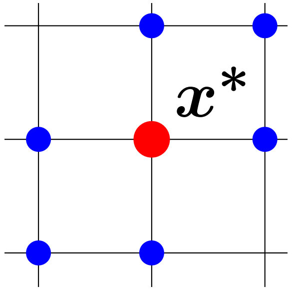
S3: Update $x^* := x^\bullet$; go to S1

What is neighborhood?

Neighborhood for Local Optimality

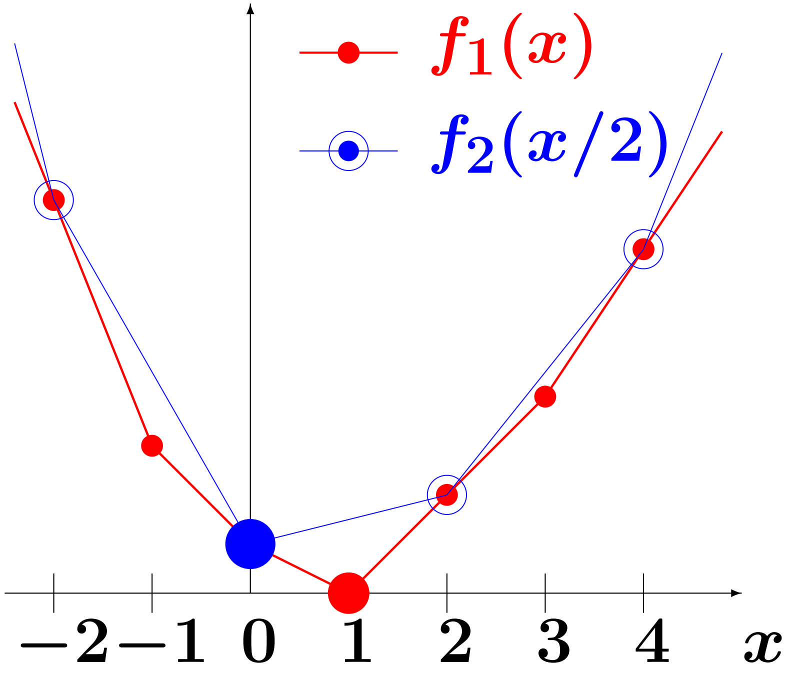
separable convex	integrally convex	L$^\natural$-convex	M$^\natural$-convex
			
$2n+1$	3^n	$2^{n+1}-1$	$n(n+1)+1$
$\{\pm e_1, \ldots, \pm e_n\}$	$\{\chi_X - \chi_Y\}$	$\{\pm \chi_X\}$	$\{e_i - e_j\}$

Local Optimality



	#neighbors	poly-time algorithm local opt	poly-time algorithm global opt
submodular (set fn)	2^n	Y	
separable-conv	$2n$	Y	
integrally-conv	3^n	N	
L$^\natural$-conv ($\mathbb{Z}^n$)	2^n	Y	
M$^\natural$-conv ($\mathbb{Z}^n$)	n^2	Y	

Scaling and Proximity



Proximity theorem:

True minimum ● exists

in a neighborhood of

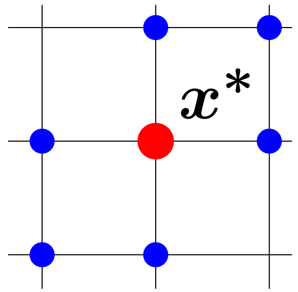
a scaled local minimum ●

$\Rightarrow$ efficient algorithm

Facts in DCA:

- Scaling preserves L-convexity
- Scaling does NOT preserve M-convexity
- Proximity thms known for L-conv and M-conv

Minimization



	#neigh -bors	poly-time algorithm local opt	global opt
submodular (set fn)	2^n	Y	Y
separable-conv	$2n$	Y	Y
integrally-conv	3^n	N	N
L$^\natural$-conv ($\mathbb{Z}^n$)	2^n	Y	Y
M$^\natural$-conv ($\mathbb{Z}^n$)	n^2	Y	Y

A2.

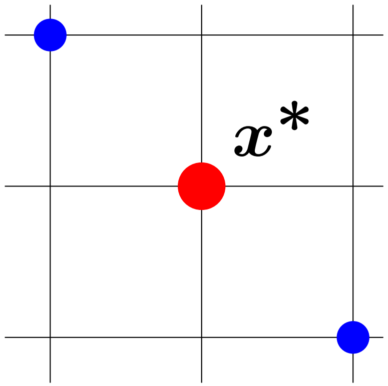
M-convex Minimization

Local vs Global Opt (M-conv)

Thm : $f : \mathbb{Z}^n \to \mathbb{R}$ **M-convex** (Murota 96)

x^*: global opt

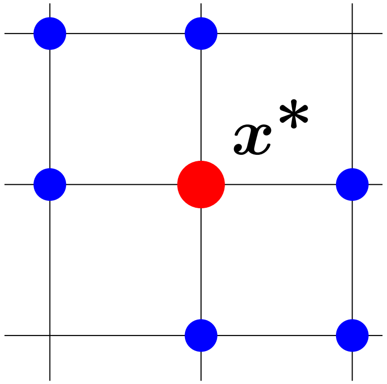
$\Longleftrightarrow$ local opt $f(x^*) \le f(x^* - e_i + e_j)$ $(\forall i, j)$



Ex: $x^* + (0,1,0,0,-1,0,0,0)$

Can check with n^2 fn evals

For M$^\natural$-convex fn $\Rightarrow$



Steepest Descent for M-convex Fn

S0: Find a vector $x \in \mathrm{dom}\, f$

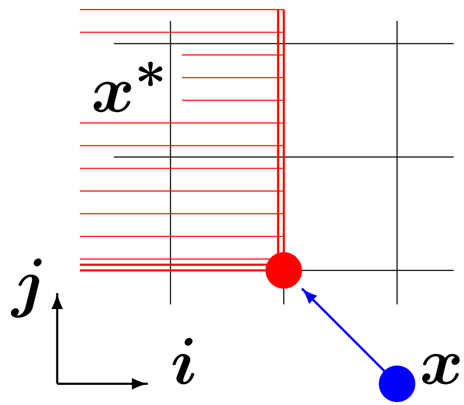
S1: Find (i, j) that $\boxed{\text{minimizes } f(x - e_i + e_j)}$

S2: If $f(x) \le f(x - e_i + e_j)$, stop

(x: minimizer)

S3: Set $x := x - e_i + e_j$

and go to S1



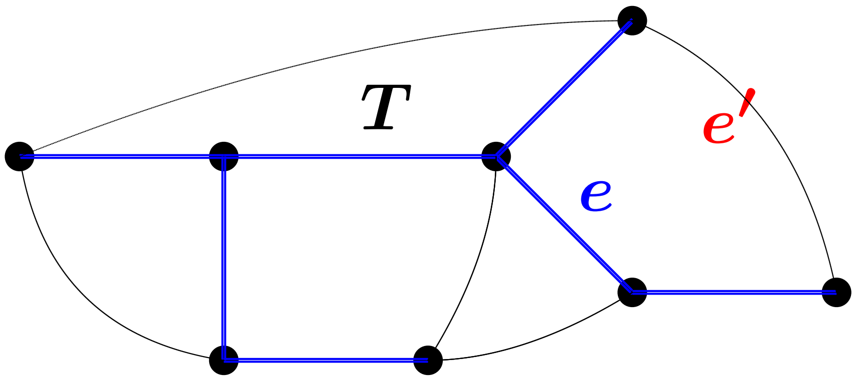
Minimizer Cut Thm (Shioura 98)

$\exists$ minimizer x^* with $x^*(i) \le x(i) - 1, \ \ x^*(j) \ge x(j) + 1$

$\Rightarrow$ Murota 03, Shioura 98, 03, Tamura 05

- Dress–Wenzel's alg for valuated matroid
- Kalaba's alg for min spanning tree

Min Spanning Tree Problem



edge length $d: E \to \mathbb{R}$

total length of T

$$\tilde{d}(T) = \sum_{e \in T} d(e)$$

Thm

T: MST $\iff \tilde{d}(T) \leq \tilde{d}(T - e + e')$

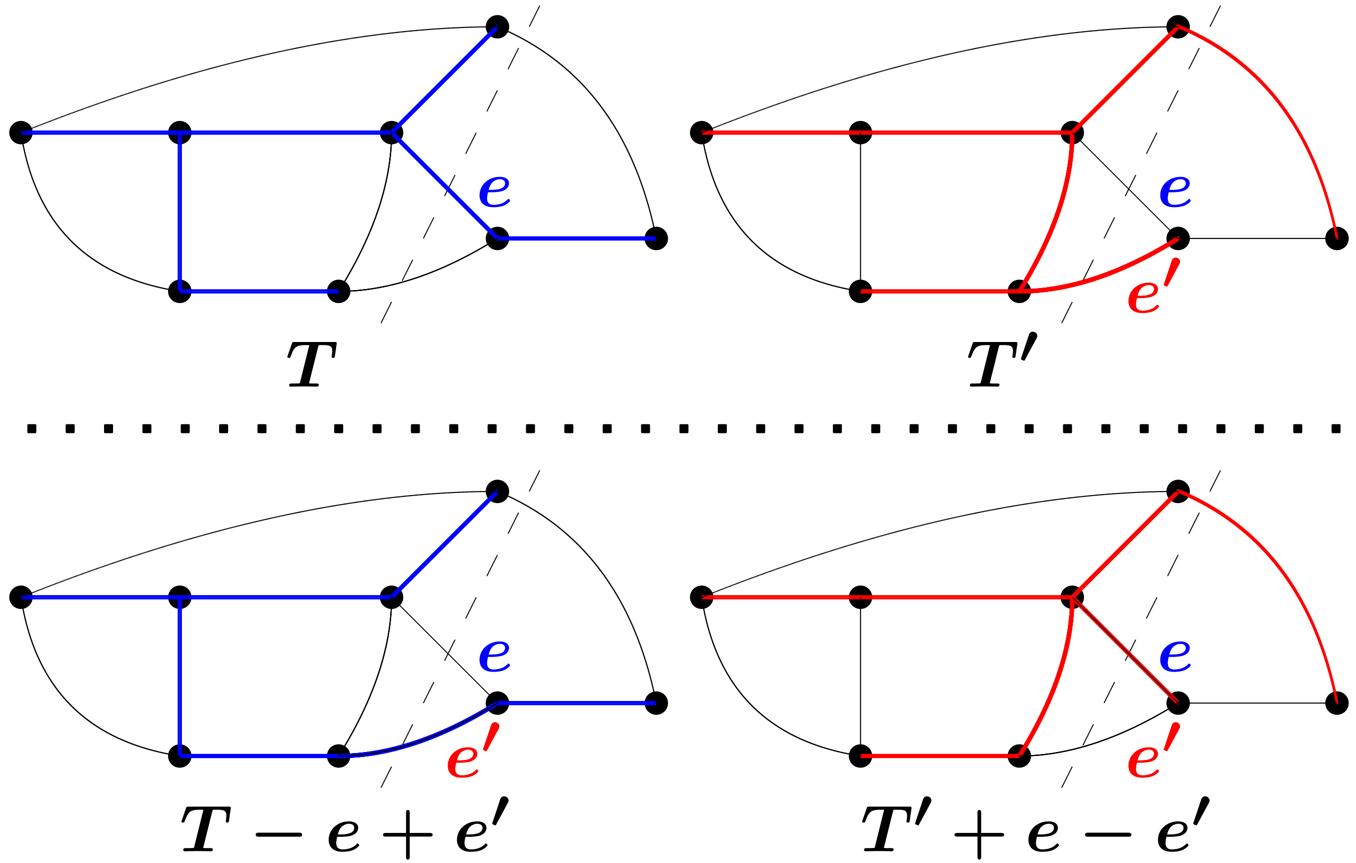
$\iff d(e) \leq d(e')$ if $T - e + e'$ is tree

Algorithm Kruskal's, Kalaba's

DCA view

- linear optimization on an M-convex set
- M-optimality: $f(x^*) \leq f(x^* - e_i + e_j)$

Tree: Exchange Property



Given pair of trees

New pair of trees

Exchange property: For any $T, T' \in \mathcal{T}$, $e \in T \setminus T'$ there exists $e' \in T' \setminus T$ s.t. $T - e + e' \in \mathcal{T}$, $T' + e - e' \in \mathcal{T}$

Kruskal's Greedy Algorithm for MST

Kruskal (1959)

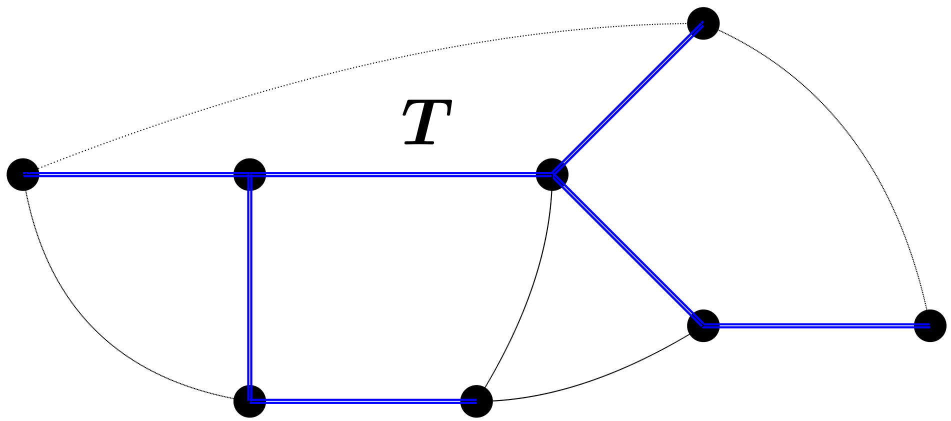
S0: Order edges by length: $d(e_1) \leq d(e_2) \leq \cdots$

S1: $T = \emptyset$; $i = 1$

S2: Pick edge e_i

S3: If $T + e_i$ contains a cycle, discard e_i

S4: Update $T = T + e_i$; $i = i + 1$; go to S2



Kalaba's Algorithm for MST

Kalaba (1960), Dijkstra (1960)

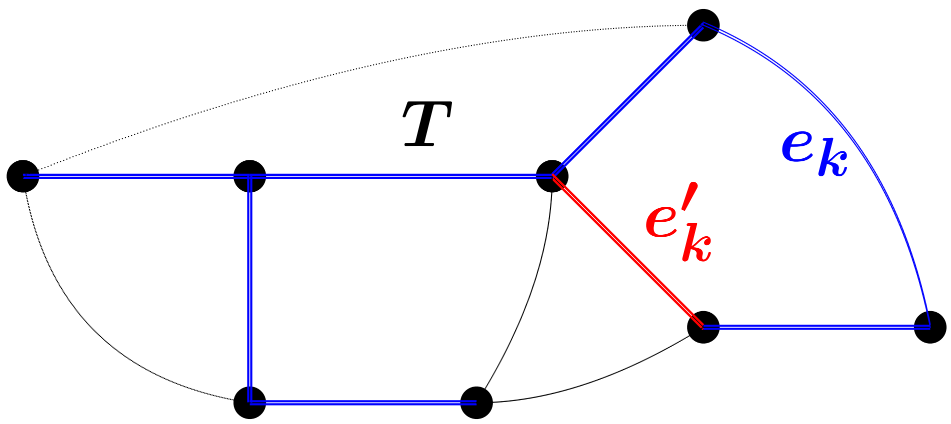
S0: $T =$ any spannning tree

S1: Order $e' \notin T$ by length: $d(e'_1) \leq d(e'_2) \leq \cdots$

$k = 1$

S2: $e_k =$ longest edge s.t. $T - e_k + e'_k$ is tree

S3: $T = T - e_k + e'_k$; $k = k + 1$; go to S2



A3.

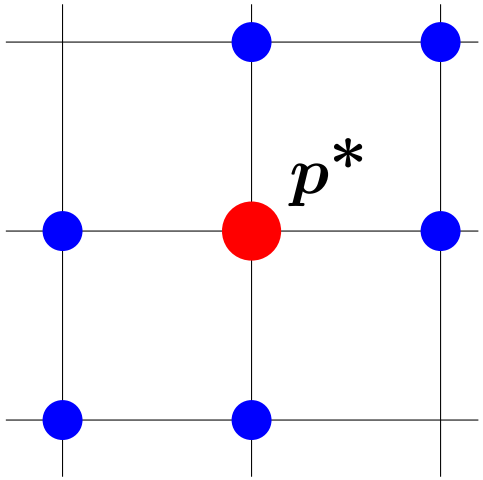
L-convex Minimization

Local vs Global Opt (L$^\natural$-conv)

Thm : $g : \mathbb{Z}^n \to \mathbb{R}$ L$^\natural$-convex (Murota 98,03)

p^*: global opt

$\iff$ local opt $g(p^*) \leq g(p^* \pm q)$ $(\forall q \in \{0,1\}^n)$



Ex: $p^* + (0,1,0,1,1,1,0,0)$

$\iff$ $\rho_{\pm}(X) = g(p^* \pm \chi_X) - g(p^*)$ takes min at $X = \emptyset$

Can check with n^5 (or less) fn evals using submodular fn min algorithm (Iwata-Fleischer-Fujishige, Schrijver, Orlin, Lee-Sidford-Wong)

Steepest Descent for L$\natural$-convex Fn

(Murota 00, 03, Kolmogorov-Shioura 09, Murota-Shioura 14)

S0: Find a vector $p^{\circ} \in \mathrm{dom}\, g$ and set $p := p^{\circ}$

S1: Find $\varepsilon = \pm 1$ and X that minimize $g(p + \varepsilon \chi_X)$

S2: If $g(p) \leq g(p + \varepsilon \chi_X)$, stop ($p$: minimizer)

S3: Set $p := p + \varepsilon \chi_X$ and go to S1

Thm: (Murota-Shioura 14)

Termination exactly in $\mu(p^{\circ}) + 1$ iterations, where

$$\mu(p^{\circ}) = \min\{\|p^* - p^{\circ}\|_{\infty}^{+} + \|p^* - p^{\circ}\|_{\infty}^{-} \mid p^* \in \arg\min g\}$$

$$\|q\|_{\infty}^{+} = \max_i \max(0, q(i)), \qquad \|q\|_{\infty}^{-} = \max_i \max(0, -q(i))$$

Monotone Steepest Descent for L$^\natural$-convex Fn

S0: Find a vector $p^\circ \in \mathrm{dom}\, g$ s.t

$\{q \mid q \geq p^\circ\} \cap \mathrm{argmin}\, g \neq \emptyset$ and set $p := p^\circ$

S1: Find X that minimizes $g(p + \chi_X)$

S2: If $g(p) \leq g(p + \chi_X)$, stop (p: minimizer)

S3: Set $p := p + \chi_X$ and go to S1

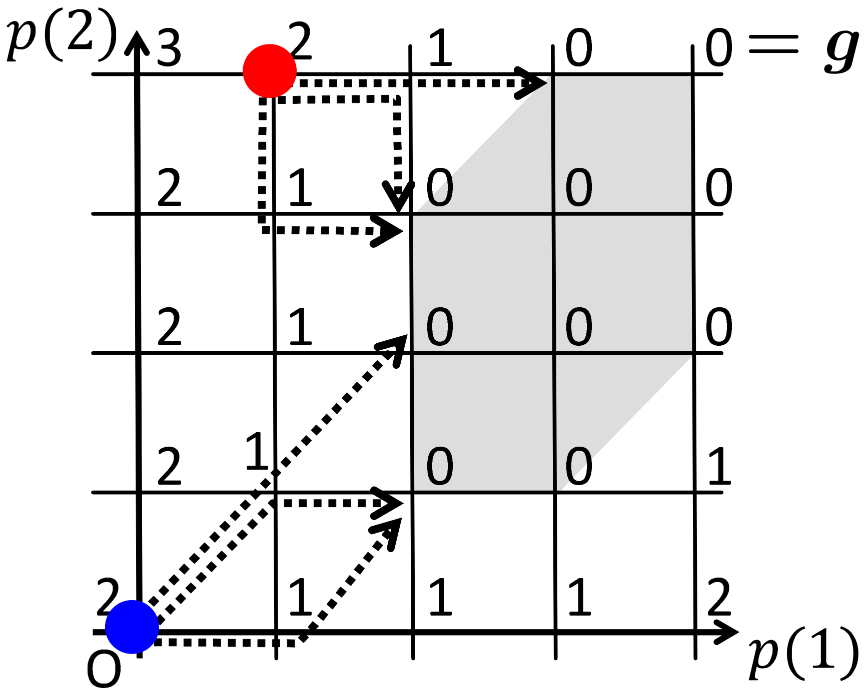
Thm: (Murota-Shioura 14)

Termination exactly in $\hat{\mu}(p^\circ) + 1$ iterations, where

$$\hat{\mu}(p^\circ) = \min\{\|p^* - p^\circ\|_\infty \mid p^* \in \arg\min g,\ p^* \geq p^\circ\}$$

$\Rightarrow$ Application to ascending auction

Steepest Descent Path for L$^\natural$-convex Fn



$$\mu(\boldsymbol{p}^\circ) = \hat{\mu}(\boldsymbol{p}^\circ) = \mathbf{2}$$

$$\|\boldsymbol{p}^\circ, \mathbf{argmin}\, \boldsymbol{g}\|_\infty = \mathbf{1}$$

$$\mu(\boldsymbol{p}^\circ) = \hat{\mu}(\boldsymbol{p}^\circ) = \mathbf{2}$$

$$\|\boldsymbol{p}^\circ, \mathbf{argmin}\, \boldsymbol{g}\|_\infty = \mathbf{2}$$

Shortest Path Problem (one-to-all)

one vertex (s) to all vertices, length $\ell \geq 0$, integer

Dual LP

Maximize	$\Sigma\ p(v)$
subject to	$p(v) - p(u) \leq \ell(u,v)\ \forall(u,v)$
	$p(s) = 0$

Algorithm Dijkstra's

DCA view

- linear optimization on an $\mathrm{L}^\natural$-convex set (in polyhedral description)
- Dijkstra's algorithm (Murota-Shioura 12)
  = steepest ascent for $\mathrm{L}^\natural$-concave maximization
  with uniform linear objective $(1, 1, \ldots, 1)$

Optimality & Proximity Theorems

Func Class	Optimality	Proximity
L-convex	$f(x^*) \le f(x^* + \chi_S) \quad (\forall S)$ $f(x^* + 1) = f(x^*)$ (M. 01)	$\|x^* - x^\alpha\| \le (n-1)(\alpha-1)$ (Iwata-Shigeno 03)
M-convex	$f(x^*) \le f(x^* - \chi_u + \chi_v)$ $(\forall u, v \in V)$ (M. 96)	$\|x^* - x^\alpha\| \le (n-1)(\alpha-1)$ (Moriguchi-M.-Shioura 02)
L2-convex (L□L convol)	$f(x^*) \le f(x^* + \chi_S) \ (\forall S)$ $f(x^* + 1) = f(x^*)$	$\|x^* - x^\alpha\| \le 2(n-1)(\alpha-1)$ (M.-Tamura 04)
M2-convex (M+M)	$f(x^*) \le f(x^* - \chi_U + \chi_W)$ $(\forall U, W;\ \|U\| = \|W\|)$ (M. 01)	$\|x^* - x^\alpha\| \le \frac{n^2}{2}(\alpha-1)$ (M.-Tamura 04)
integrally convex	$f(x^*) \le f(x^* - \chi_U + \chi_W)$ $(\forall U, W)$ (Favati-Tardella 90)	$\|x^* - x^\alpha\| \le \frac{(n+1)!}{2^{n-1}}(\alpha - 1)$ (Moriguchi-M.-Tamura -Tardella 16)

$\| \cdot \| = \| \cdot \|_\infty$

A4.
M-convex Intersection

(Fenchel Duality)

Intersection Problem $(f_1 + f_2)$

Recall: $\mathrm{L}^\natural + \mathrm{L}^\natural \Rightarrow \mathrm{L}^\natural$, $\mathrm{M}^\natural + \mathrm{M}^\natural \not\Rightarrow \mathrm{M}^\natural$

M-convex Intersection Algorithm:

Minimizes $f_1 + f_2$ for $\mathrm{M}^\natural$-convex f_1, f_2

$\Leftrightarrow$ Maximizes $f_1 + f_2$ for $\mathrm{M}^\natural$-concave f_1, f_2
(submodular function maximization)

$\Leftrightarrow$ Fenchel duality (min = max)

$\Rightarrow$ Valuated matroid intersection (Murota 96)

$\Rightarrow$ Weighted matroid intersection

(Edmonds, Lawler, Iri-Tomizawa 76, Frank 81)

M-convex Intersection: Min [M♮+M♮]

M♮+M♮ is NOT M♮

$f_1,\ f_2$: M♮-convex $(\mathbb{Z}^n \to \mathbb{R})$, $x^* \in \mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2$

(1) x^* minimizes $\boxed{f_1 + f_2}$ (Murota 96)

$\iff$ $\exists\, p$ (certificate of optimality)

- x^* minimizes $\boxed{f_1(x) - \langle p, x \rangle}$ (M-opt thm)
- x^* minimizes $\boxed{f_2(x) + \langle p, x \rangle}$ (M-opt thm)

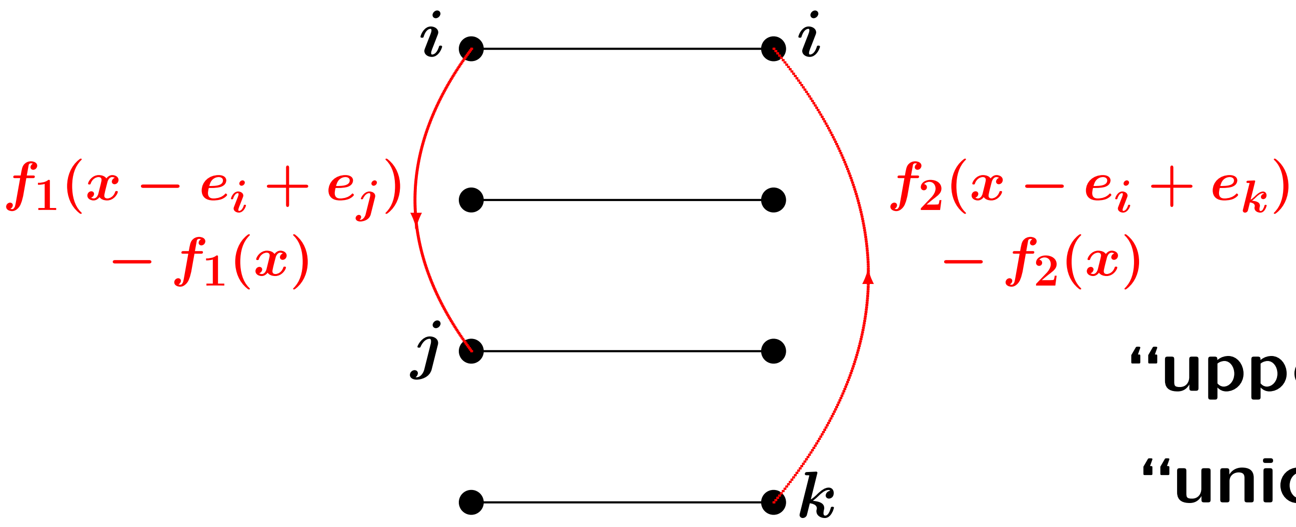
(2) $\mathrm{argmin}\,(f_1 + f_2) = \mathrm{argmin}\,(f_1 - p) \cap \mathrm{argmin}\,(f_2 + p)$

(3) $f_1,\ f_2$ are integer-valued $\Rightarrow$ integral p

M-convex Intersection Algorithms

Natural extensions of
weighted (poly)matroid intersection algorithms

Exchange arcs are weighted



"upper-bound lemma"

"unique-max lemma"

- cycle-canceling (Murota 96, 99)
- successive shortest path (Murota-Tamura 03)
- scaling (Iwata-Shigeno 03, Iwata-Moriguchi-Murota 05)

Convolution

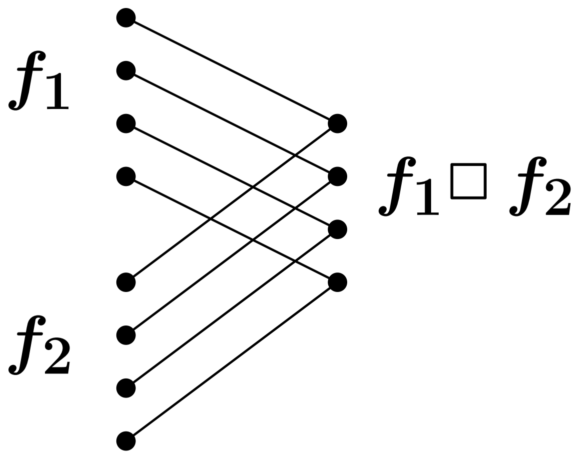
Convolutions of M$^\natural$-convex functions:

$$(f_1 \Box f_2)(x) = \min_y (f_1(y) + f_2(x-y))$$

$$(f_1 \Box f_2 \Box f_3)(x), \quad (f_1 \Box f_2 \Box \cdots \Box f_k)(x)$$

can be computed by M-convex intersection algorithms

cf. aggregated utility function



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